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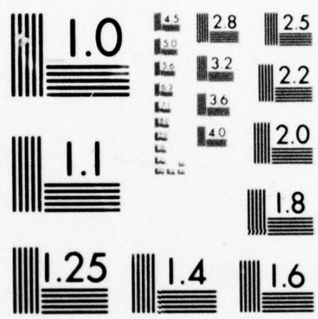
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Numerical Investigation of the Effect of a Coriolis Force on the Stability of Plane Poiseuille Flow

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by

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ABSTRACT

The stability of the viscous flow between two parallel horizontal plates due to a constant reduced pressure gradient in a system rotating about a vertical axis is studied. The critical value of the Reynolds number R, based on the reduced pressure gradient, is a function of a dimensionless rotation parameter T, the Taylor number. A numerical solution of the eigenvalue problem shows that (i) the viscous instability mode associated with plane Poiseuille flow at $T = 0$ disappears at a value of T of about $T \cong 0.4$, and (ii) for $T \neq 0$ a new instability mode appears as a result of the Coriolis effect on the basic flow and in the perturbation equations. This new instability gives the critical value of R for values of T as small as 0.06.

I INTRODUCTION

The Navier-Stokes equations have an exact solution for the flow between two horizontal parallel planes due to a pressure gradient in the presence of rotation about an axis perpendicular to the planes. The rotation forces a component of flow in a horizontal direction perpendicular to the direction of the pressure gradient as well as a modification of the parallel component. Wollkind and DiPrima[1] studied the stability of this flow for small values of the dimensionless rotation parameter T [see Eq. (1)] by means of an expansion in T. In this paper we use direct numerical procedures to solve the stability problem for moderate values of T.

The results confirm those of Wollkind and DiPrima for $T \to 0$ for the instability of the critical mode associated with the stability of plane Poiseuille for $(T = 0)$. The calculations also show the rather unexpected result (and one not obtainable by the expansion procedures used by Wollkind and DiPrima) that this viscous-driven instability mode disappears at a value of T slightly greater than 0.35. Moreover, they reveal the presence of a second type of instability when $T \neq 0$. This new instability gives the critical Reynolds number of the basic flow for values of T greater than about 0.06. The critical disturbance is a wave propagating very slowly (almost a standing wave) at an angle (nearly perpendicular for T very small) to the direction of the pressure gradient.

The stability problem is formulated in Sec. II and the numerical procedure for solving the eigenvalue problem is explained in Sec. III. Results given in Sec. IV are summarized in Sec. V.

II THE STABILITY PROBLEM

The reader is referred to Wollkind and DiPrima[1] for the details of the formulation of the stability problem. We present here only a summary sufficient for us to state the stability problem.

Consider an xyz coordinate system which is rotating about the z axis with constant angular velocity Ω. We suppose that there are planes at $z = 0$ and $z = d$, and that there is a constant reduced pressure gradient $\partial p/\partial x$ in the direction of the positive x axis. Let

$$T = \frac{\Omega d^2}{4\nu}, \qquad U^* = \frac{-d^2}{8\rho\nu}\frac{\partial p}{\partial x}, \qquad R = \frac{U^* d}{2\nu}, \tag{1}$$

where ν and ρ are the kinematic viscosity and density, respectively. The parameter T, the Taylor number, is a measure of the ratio of the Coriolis force to the viscous force. It is the recipient of the Ekman number. The parameter U^* is an appropriate velocity scale for the case $\partial p/\partial x$ = constant (assumed negative so that $U^* > 0$), and R is the Reynold number.

An exact solution of the equations of motion satisfying the no slip boundary conditions at the walls is

$$U_B = T^{-1}[-\sinh\zeta \sin\zeta + A\cosh\zeta \sin\zeta + B\sinh\zeta \cos\zeta], \tag{2}$$

$$V_B = T^{-1}[\cosh \zeta \cos \zeta - 1 - A \sinh \zeta \cos \zeta + B \cosh \zeta \sin \zeta], \tag{3}$$

$$\begin{aligned} A &= [\sinh (2T^{1/2})]/[\cosh (2T^{1/2}) + \cos (2T^{1/2})] \quad , \\ B &= [\sin (2T^{1/2})]/[\cosh (2T^{1/2}) + \cos (2T^{1/2})] \quad , \end{aligned} \tag{4}$$

where $\zeta = T^{1/2}z$ and z is now a dimensionless variable scaled with respect to d/2 so that $0 \leq z \leq 2$. The dimensionless velocities U_B and V_B (with respect to U*) in the x and y directions, respectively, are symmetric about the center of the channel (z = 1). For T small the velocity reduces to a small perturbation from plane Poiseuille flow: $U_B = z(2-z) + O(T^2)$ and $V_B = T(-z^4 + 4z^3 - 8z)/6 + O(T^3)$. For T large and z bounded away from the walls, we obtain the geostrophic flow $U_B \sim 0$ and $V_B \sim (\partial p/\partial x)/2\rho\Omega U^* = -T^{-1}$. We note, since $\partial p/\partial x < 0$ so that $U^* > 0$, that the cross flow is in the negative y direction when the rotation is counterclockwise.

Next we assume that all quantities have been made dimensionless: lengths with respect to d/2, velocities with respect to U*, and time with respect to d/2U*. To study the stability of the basic flow (U_B, V_B) we superimpose a small disturbance, (i) substitute in the equations of motion and neglect quadratic terms, (ii) look for normal mode solutions of the form

$$u(x,y,z,t) = U_B(z) + u'(z) \exp[i(\alpha x + \beta y - kct)] \ , \tag{5}$$

where $k = (\alpha^2 + \beta^2)^{1/2}$, (iii) introduce new coordinates (x_1, y_1) with velocity perturbation amplitudes u_1 and v_1, respectively, by rotating the (x,y) coordinate system through an angle θ measured counterclockwise from the positive x direction so that $x_1 = x \cos \theta + y \sin \theta$, $y_1 = -x \sin \theta + y \cos \theta$, where $\cos \theta = \alpha/k$

and $\sin\theta = \beta/k$, (iv) multiply the x-momentum equation by $(-\alpha)$, the y-momentum equation by β and add to obtain a new equation, (v) multiply the x-momentum equation by $(-\beta)$, the y-momentum equation by α and add to obtain a second equation, and (vi) eliminate the pressure by using the z-momentum equation, and use the continuity equation to introduce a function ϕ_1 such that $u_1 = -d\phi_1/dz$ and $w' = ik\phi_1$.

The result of these calculations is the following sixth order system of ordinary differential equations:

$$(D^2 - k^2)^2\phi_1 - ikR[(U_1-c)(D^2-k^2)\phi_1 - (D^2U_1)\phi_1] - 2TDv_1 = 0, \tag{6}$$

$$(D^2 - k^2)v_1 - ikR[(U_1-c)v_1 + (DV_1)\phi_1] + 2TD\phi_1 = 0 . \tag{7}$$

The velocities U_1 and V_1 are given by

$$U_1(z,T,\theta) = U_B(z,T)\cos\theta + V_B(z,T)\sin\theta \quad , \tag{8}$$

$$V_1(z,T,\theta) = -U_B(z,T)\sin\theta + V_B(z,T)\cos\theta \quad , \tag{9}$$

and in the (x_1,y_1,z) coordinate system the perturbation is proportional to $\exp[ik(x_1-ct)]$.

The boundary conditions are

$$\phi_1 = D\phi_1 = v_1 = 0 \text{ at } z = 0 \quad , \tag{10}$$

and for disturbances for which ϕ_1 is symmetric and v_1 is antisymmetric about $z = 1$,

$$D\phi_1 = D^3\phi_1 = v_1 = 0 \text{ at } z = 1. \tag{11}$$

For disturbances for which ϕ_1 is anti-symmetric and v_1 is symmetric about $z = 1$, the boundary conditions would be $\phi_1 = D^2\phi_1 = Dv_1 = 0$ at $z = 1$. It is known for $T = 0$ that the critical disturbance is one for which ϕ_1 is symmetric about $z = 1$, and only such disturbances will be considered here.

Equations (6), (7), (10), and (11) define an eigenvalue problem of the form $F(T,R,\theta,k,c) = 0$. The Taylor number T, the Reynolds number R, and the angle of propagation θ are real. For temporally growing or decaying disturbances, the wavenumber k is real and the parameter c (the complex amplification rate) is complex, $c = c_r + ic_i$. For spatially growing or decaying disturbances k is complex with kc real. In this paper we consider the case of temporal instability. Thus, k is real and the flow is unstable if there exists an eigenvalue c such that $c_i > 0$. For a given value of T the condition that the eigenvalue with largest imaginary part have $c_i = 0$ determines a neutral surface in R,θ,k space which separates stable from unstable states. The critical value of R is the smallest value of R that corresponds to a point on the neutral surface: $R_c(T) = \min R(T,k,\theta,c_r,c_i = 0)$ for $k > 0$ $-\pi/2 < \theta \leq \pi/2$. (Note that θ can be restricted to this interval since c_r can be positive or negative.) The corresponding values of k,θ,c_r determine the wavenumber, direction of propagation, and the wave velocity of the critical disturbance.

Before turning to the numerical procedure for solving the eigenvalue problem, it is useful to derive an "energy" integral associated with Eqs. (6) and (7). Multiplying Eqs. (6) and (7) by the complex conjugates of ϕ_1 and v_1, respectively, integrating

by parts, adding, and taking the real part, we obtain

$$kRc_i(E_\phi + E_v) = -(D_\phi + D_v) + R(M + H) \quad . \tag{12}$$

Here

$$E_\phi = \int_0^1 (|D\phi_1|^2 + k^2|\phi_1|^2)dz \ , \quad E_v = \int_0^1 |v_1|^2 dz \ ,$$

$$D_\phi = \int_0^1 |(D^2-k^2)\phi_1|^2 dz \ , \quad D_v = \int_0^1 (|Dv_1|^2 + k^2|v_1|^2)dz \ , \tag{13}$$

$$M = k\int_0^1 (DU_1)(\phi_{1r}D\phi_{1i}-\phi_{1i}D\phi_{1r})dz \ , \quad H = k\int_0^1 (DV_1)(\phi_{1i}v_{1r}-\phi_{1r}v_{1i})dz \ .$$

The quantity $kRc_i(E_\phi + E_v)$ can be interpreted as the rate of change of disturbance kinetic energy, $D_\phi + D_v$ is the rate of disturbance viscous dissipation, RM is the rate of transfer of kinetic energy from the U_1 component of the basic velocity to the disturbance, and RH is the rate of transfer of kinetic energy from the V_1 component of the basic velocity to the disturbance.

Since E_ϕ, E_v, D_ϕ, D_v are positive definite, it follows that M and/or H must be positive if disturbances are to grow ($c_i > 0$). If M is positive, disturbance energy is gained from the U_1 component of the basic velocity which is parallel to the line of propagation of the disturbance, while if H is positive, disturbance energy is gained from the V_1 component of the basic velocity which is perpendicular to the line of propagation of the disturbance. It is

interesting that T does not appear explicitly in Eq. (12); it enters only through its effect on the basic velocity field and the solution of the perturbation equations.

III NUMERICAL PROCEDURE

It is well known that numerical solutions of Orr-Sommerfeld like problems are difficult to obtain, because of the presence of solutions that grow very rapidly in z when the Reynolds number is large. The difficulties can largely be overcome by using orthogonalization techniques (cf., Conte[2], Davey[3], and Scott and Watts[4]). For the problem under consideration, we followed Davey's method, which uses parallel shooting and orthonormalizations to solve the system (6) to (11) for prescribed values of T, R, θ, k and an initial guess for the eigenvalue c. Successive iterates for c were obtained using Muller's method[5] to find a root of the transcendental equation $F(T,R,\theta,k,c) = 0$.

The differential equations (6) and (7) were integrated using an implicit Adams method with a step size of 1/250 and four orthonormalizations for Reynolds numbers of approximately 6000. In order to estimate the accuracy of the numerical solutions, we (i) calculated several eigenvalues for plane Poiseuille flow and compared results with Davey[3], (ii) calculated solutions of a model sixth order problem having a known exact solution, and (iii) calculated residuals of the differential equations (6) and (7) using the computed values of ϕ_1, v_1, and c. In general, we estimate that the eigenvalues and eigenfunctions are accurate to six or seven significant digits.

Our primary goal was to find points on the neutral surface. Thus, once a satisfactory eigenvalue c was found by Davey's procedure, we changed R and/or θ and used linear extrapolation to search for a point on the neutral surface. An attempt was made to change R and/or θ so that the search direction was approximately orthogonal to the neutral surface.

Once a point on the neutral surface was found, the corresponding eigenvector (ϕ_1, v_1) was calculated using a technique similar to those used by Conte[2] and Scott and Watts[4]. The entire process worked satisfactorily provided a sufficiently close initial guess for a point on the neutral surface was known. This was generally easy to accomplish by continuation from results already obtained. Typically, four to six Muller iterations were required to find an eigenvalue and three or four linear extrapolations were required to find a point on the neutral surface; thus, the differential equations (6) and (7) had to be solved from 12 to 24 times for each point on the neutral surface.

Finally, minimum points on the neutral surface were obtained by adjusting k and/or θ using a pattern search procedure (cf., Jacoby, Kowalik, and Pizzo[6], Chapter 4) to locate the minimum value of R for a fixed value of T. This procedure also worked quite well provided that relatively small search steps were used.

IV RESULTS

For $T = 0$ it is known (Squire's Theorem[7]) that the critical Reynolds number for the eigenvalue problem (6)-(11) occurs for $\theta = 0$. The values given by Reynolds and Potter[8] are

$$\theta_c = 0 \ , \quad k_c = 1.0207 \ , \quad R_c = 5772.12 \ , \quad c_{rc} = 0.26402. \tag{14}$$

Wollkind and DiPrima[1] found for small T that this critical point was perturbed as follows:

$$\begin{aligned} \theta_c &= -2.044T + O(T^3) \ , \\ k_c &= 1.0207 + O(T^2) \ , \\ R_c &= 5772.12[1 - 0.3877T^2 + O(T^4)] \ , \\ c_{rc} &= 0.26403 + O(T^2) \ . \end{aligned} \tag{15}$$

They noted that these results could only be expected to be valid for very small T and suggested that they are probably not valid at $T = 0.5$.

We used the method described in Sec. III to obtain k_c, θ_c, R_c, and c_{rc} for $T = 0.05$, 0.1, 0.2, 0.3, 0.4, 0.5, 0.75, and 1.0. The results are given in Table 1. Observe that for $T = 0.05$, 0.1, 0.2, and 0.3 there are two relative minima: one near $\theta = 0$ and $k = 1.02$ and one near $\theta = 1.3$ to 1.5 and $k = 1.37$. The first of these, which we denote by (I), is clearly a small perturbation due to rotation of the critical parameters for the instability of plane Poiseuille flow ($T = 0$). The perturbation formulas (15) give results in reasonable agreement with the numerical calculations

of the critical values for the Type I disturbance. The comparison is shown in Table 2. The second minimum, which we denote by (II), is a new instability introduced by the rotation; it will be discussed later in this section. We note that two relative minima on a neutral curve were found by Gill and Davey[9] in their analysis of the instability of a buoyancy driven system, and by Lilly[10] in his analysis of the instability of the Ekman boundary layer. Each of these eigenvalue problems involves a sixth order system of differential equations that are similar to Eqs. (6) and (7); however, the domain for each problem is $(0,\infty)$ in contrast to the finite domain of the present problem.

The two relative minima of the neutral surface are shown in Fig. 1 where the variation of R on the neutral surface is given as a function of θ for $k = 1.02$ for the cases $T = 0$, 0.05, and 0.2. Similar curves are given in Fig. 2 for $T = 0.3$ and 0.5. The minimum point corresponding to the type I instability disappears at a value of T between 0.3 and 0.5 as can be seen from Fig. 2 (it was not possible to locate a minimum at $T = 0.4$). The width of the valley associated with the type II instability is extremely narrow for very small values of T, but increases as T increases. The curves in Figs. 1 and 2 are incomplete. While we were able to take our calculations up to Reynolds numbers as large as 500,000 and obtain (θ,R) points not shown in Figs. 1 and 2, we were not able to determine if the curves have a finite maximum or go to infinity (as is the case for $T = 0$, $\theta \rightarrow \pm \pi/2$).

The corresponding variation of c_r with θ for $k = 1.02$ and $T = 0$, 0.5, and 0.2 is shown in Fig. 3. We note that since R is a periodic function of θ with period π it follows that $c_r(\theta=\pi/2) = - c_r(\theta=-\pi/2)$.

It is clear from Table 1 and Fig. 1 that the absolute minimum on the neutral surface as $T \to 0$ corresponds to a type I instability. However, the type II instability rapidly becomes dominant and gives the absolute minimum at a value of T as small as 0.06 (approximately). The variation of R_c, c_{rc}, θ_c, and k_c with T is shown in Figs. 4, 5, 6, and 7. The variation of R_c with T is, of course, continuous; however, each of the other parameters change discontinuously when the type I instability is replaced by the type II instability.

The eigenfunctions corresponding to the parameter values for each of the two relative minima for T = 0.05 are shown in Figs. 8 and 9. The normalization for these eigenfunctions as well as the case shown in Fig. 10 is $\phi_1(1) = 1$. The ϕ_1 component of the eigenfunction for the type I instability for T = 0.05 (Fig. 8a and 8b) is very similar to the corresponding eigenfunction for the instability of plane Poiseuille flow (T = 0); for example, see Fig. 3a of Reynolds and Potter[8], and for a larger value of R see Fig. 1a of Lee and Reynolds[11]. Note the relative simplicity and the different order of magnitudes of the eigenfunction for the type II instability (especially the v_1 component) compared to the eigenfunction for the type I instability. The eigenfunction associated with the critical parameter values on the neutral surface for T = 1.0 is shown in Fig. 10. The simplicity of the type II instability eigenfunction suggests the instability might be calculated by procedures such as the Galerkin method; however, we have not pursued this idea.

We turn now to the type II instability which determines the critical Reynolds number for T > 0.06 (approximately). Notice from Table 1 that the value of c_r is extremely small and positive;

hence the disturbance is almost a standing wave moving very slowly in the direction given by θ_c. For small T we have $\theta_c \cong \pi/2$; hence the direction of propagation is nearly perpendicular (positive y direction) to the direction of the basic flow when T = 0 (the positive x direction). Moreover, we note that the positive y direction is opposite to the direction of the geostrophic flow (negative y direction) that occurs when T is large. With increasing T the angle of the direction of propagation decreases from $\pi/2$, and at T = 1 we have $\theta_c = 0.86$ rad $\cong 50°$.

There are two ways in which the Coriolis effect manifests itself in the stability analysis. Firstly, it modifies the basic velocities U_1 and V_1 given by Eqs. (8), (9), (2), and (3) that appear in the distrubance equations (6) and (7); secondly, it couples Eqs. (6) and (7) through the terms $-2TDv_1$ in the equation for ϕ_1 and the term $2TD\phi_1$ in the equation for v_1.

It is well known that the eigenvalue problem for the Orr-Sommerfeld equations [Eq. (6) with T = 0] is very sensitive to the basic velocity profile U_1 and that inflection points play a special role. For T = 0.2 the velocity profiles $U_1(z,\theta)$ are shown in Fig. 11 for several values of θ in the neighborhood of $\theta_c = 1.404$. It is clear from Fig. 11 that $U_1(z,\theta)$ has an inflection point for $\theta = 1.404$ and for nearby values of θ. Indeed, a simple analysis shows that $U_1(z,\theta)$ first exhibits an inflection point at z = 1 when $\theta = \tan^{-1}(\cot\sqrt{T}\,\coth\sqrt{T})$, and that the inflection point decreases from z = 1 to z = 0 where it disappears when θ has increased to $\pi/2$.

The interval in θ for which $U_1(z,\theta)$ has an inflection point corresponds almost exactly to the valley near $\theta = \pi/2$ in the R vs. θ curve shown in Fig. 1. For other small values of T the situation is the same: the velocity profile $U_1(z,\theta)$ has an inflection point for each value of θ in a small interval about the corresponding value of θ_c for the type II instability.

This suggests that the type II instability is an inviscid instability associated with the inflection point that occurs in the U_1 component of the basic velocity profile as a result of the Coriolis force. [For $T = 0$ we have $U_1 = z(2-z)\cos\theta$ so the inflection point is a Coriolis effect.] In Table 1 we have given the value of z^* at which the inflection point occurs for θ_c and the value of $U_1(z^*,\theta_c)$. It is known if viscosity is neglected in the Orr-Sommerfeld equation ($R \to \infty$) that for neutral disturbances $U_1 - c_r$ must vanish at least once for $0 < z < 1$. Such points are critical points of the inviscid disturbance equation. For velocity profiles with a point of inflection z^* and only one critical point, we have c_r equal to the velocity at the inflection point. However, for velocity profiles with an inflection point and two critical points, which is typical of several of the velocity profiles shown in Fig. 11, Gregory, Stuart, and Walker[12] (see Section 12) have shown that c_r will be slightly less than the velocity at the inflection point. It is clear from Table 1 that $U_1(z^*,\theta_c)$ and c_r are of the same (small) size with $c_r \cong 2U_1(z^*,\theta_c)$. It is significant that c_r and $U_1(z^*,\theta_c)$ have comparable values. We have no explanation as to why $c_r = 2U_1(z^*,\theta_c)$ rather than $c_r \leq U_1(z^*,\theta_c)$; except to note that viscous and Coriolis effects are not included in this discussion.

The importance of the modification of the U_1 component of the basic velocity due to rotation can be shown as follows. Firstly, consider Eq. (6) with $T = 0$, the Orr-Sommerfeld equation. Then $U_1 = z(2-z)\cos\theta$ and at $k = 1.02$, $\theta = 1.43$ the critical value of R is $5772.12 \sec(1.43) = 41{,}132$. Now for $T = 0.2$ we neglect the Coriolis term $(-2TDv_1)$ in Eq. (6), but use the U_1 velocity profile that is correct for this value of T. Then we find that for $k = 1.02$ the minimum point on the neutral curve occurs for $\theta = 1.43$ and has the value $R = 6688$. This is a significant change from $R \approx 41{,}000$. Moreover, the Orr-Sommerfeld equation with the U_1 velocity profile corresponding to $T = 0.2$ gives the qualitative features of the full sixth order system in the neighborhood of the type II minimum point as can be seen from Fig. 12. On the other hand, modification of the basic velocity U_1 by rotation is *not the entire answer.* For the full sixth order system, Eqs. (6) and (7), the minimum point on the neutral surface for $T = 0.2$, $k = 1.02$ is $R = 1953$ at $\theta = 1.40$.

The energy integral relation (12) is also useful. In Table 3 the values of the different integrals have been calculated for the minimum points on the neutral curves corresponding to type I and type II disturbances. In these calculations, the eigenfunctions have been further normalized by the requirement $E_v = 1$. Also, the integrals were evaluated using the trapezoidal rule with an interval size of 1/50 for most of the calculations; hence the evaluation of the integrals is not as accurate as the eigenfunctions. Recall that RM and RH represent the rate of transfer of kinetic energy from the U_1 and the V_1 components of the basic velocity,

respectively, to the disturbance. Consider the type I instability. For $T = 0.05$, when the type I disturbance is the critical disturbance, the primary contribution to the disturbance energy is from the U_1 components as would be expected. However, with increasing T the situation changes rapidly and at $T = 0.3$ the primary contribution (by an order of magnitude) comes from the perpendicular component V_1 of the basic velocity.

For the type II instability, essentially all of the disturbance energy is contributed by the interaction of V_1 with ϕ_1 and v_1. It is clear from Table 3 that the energy balance for a type II neutral disturbance is $0 = -D_v + RH$. Thus, the type II instability appears to be driven by the Coriolis force through (i) modification of the basic U_1 component of velocity, and (ii) the transfer of energy through the interaction of the Reynolds stress and the gradient of the V_1 component of velocity.

Finally, we consider the direct coupling effect of the Coriolis force in Eqs. (6) and (7). When $T = 0$, Eq. (6) with the boundary conditions on ϕ_1 from Eqs. (10) and (11) can be used to determine the eigenvalue relation. Then the nonhomogeneous differential equation (7) can be solved for v_1 subject to the boundary conditions (10) and (11), provided that the corresponding homogeneous problem does not have a solution. It can readily be demonstrated as first shown by Squire[7] that the latter problem only has solutions corresponding to decaying disturbances ($c_i < 0$); hence, no difficulty will be encountered in calculating neutral or amplified disturbances.

(Nor is it likely that an eigenvalue $c_i < 0$ of the boundary value problem for ϕ_1 will be an eigenvalue of the homogeneous boundary value problem for v_1.) Indeed, this calculation forms the first term in a perturbation calculation in powers of T (see Wollkind and DiPrima[1]).

However, there is another solution when $T = 0$; namely, the solution $\phi_1 = 0$ with the eigenvalue relation determined by

$$(D^2 - k^2)v_1 - ikR(U_1 - c)v_1 = 0 \quad , \quad v_1(0) = v_1(1) = 0. \tag{16}$$

As noted above, it is not difficult to show that the eigenvalues (c) of this problem give decaying disturbances. Indeed, if we write $U_1(z,\theta) = U_{10}(z)\cos\theta$, $c = c_0\cos\theta$, and $R = R_0/\cos\theta$, where $U_{10}(z) = z(2-z)$, and then multiply the equation by the complex conjugate of v_1 and integrate, we find $0 < c_{0r} < 1$ *and*

$$c_{0i} = -\frac{1}{kR_0} \frac{\int_0^1 (|Dv_1|^2 + k^2|v_1|^2)dz}{\int_0^1 |v_1|^2 dz} \le -\frac{1}{kR_0}(\pi^2 + k^2) . \tag{17}$$

Equation (16) is for an antisymmetric solution; for a symmetric solution the π^2 in Eq. (17) is replaced by $\pi^2/4$. Thus, the symmetric v_1 mode with $\phi_1 = 0$ may be less stable than the anti-symmetric v_1 mode with $\phi_1 = 0$.

We have not tried to study how the eigenvalues of the two problems move for $T \neq 0$. This is an interesting mathematical problem whose analysis may clarify the origin of the type II instability.

V SUMMARY

The present numerical calculations show (i) the type I instability associated with the viscous instability of plane Poiseuille flow gives the critical Reynolds number for $T \to 0$, (ii) the type I instability disappears at a Taylor number of about 0.4, (iii) there is a type II instability due to Coriolis effects, (iv) the type II instability gives the critical Reynolds number for Taylor numbers slightly greater than 0.06 (approximately), and (v) the type II instability represents a wave propagating very slowly (almost a standing wave) along a direction at an angle measured counterclockwise from the direction of the applied pressure gradient that decreases from nearly 90° for $T \gtrsim 0.06$ to about 50° at $T = 1$.

It is interesting that an almost standing wave solution also occurs at one of the two relative minima on the neutral surface in the stability of the Ekman boundary layer[10] ($T \to \infty$). However, for that problem the stationary wave solution does not give the absolute minimum of the neutral surface. Rather stationary waves become unstable at a Reynolds number of about twice the critical Reynolds number for the growth of a viscous-Coriolis disturbance which propagates at a small angle from the direction of the geostrophic flow. It is tempting to conjecture that for increasing T the viscous instability mode of plane Poiseuille flow is replaced by an inflection point - Coriolis instability at $T = 0.06$ which persists as an almost standing wave solution for all T, but is replaced (in determining R_c) by a viscous-Coriolis mode for large T.

The destabilizing effect of even a small amount of rotation (small T) is clearly shown in Fig. 4; for example, the values of R_c at T = 0, 0.1, and 0.2 are 5772, 3575, and 1806, respectively, with $R_c = 468$ at T = 1. It also appears from Fig. 4 that the curve of R_c vs. T has nearly reached a minimum at T = 1, and beyond this minimum R_c will start to increase with increasing T. This would be consistent with the asymptotic value $R \sim 55T^{3/2}$ as $T \to \infty$ given by Wollkind and DiPrima[1] which is based on Lilly's[10] analysis of the stability of the Ekman boundary layer.

Finally, we note that the values of R_c obtained here for $0 \leq T \leq 1$ lie above (as they should) the energy limit for the growth of disturbances calculated by Jankowski and Squire[13].

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Table 1. Minimum points on the neutral surface

T	k	θ	R	c_r	z^*	$U_1(z^*,\theta)$
			Type I Disturbance			
0.05	1.021	-0.1026	5767	0.2635		
0.1	1.022	-0.2061	5750	0.2620		
0.2	1.027	-0.4234	5674	0.2553		
0.3	1.039	-0.6786	5519	0.2413		
0.35	1.052	-0.8577	5387	0.2263		
			Type II Disturbance			
0.04	1.367	1.537	8911	0.002208	0.607	0.00127
0.05	1.370	1.529	7132	0.002634	0.596	0.00127
0.1	1.368	1.487	3575	0.005441	0.600	0.00277
0.2	1.367	1.404	1806	0.01100	0.601	0.00567
0.3	1.369	1.324	1224	0.01594	0.599	0.00804
0.4	1.369	1.246	939.5	0.02081	0.599	0.0105
0.5	1.3697	1.1713	772.9	0.02533	0.599	0.0128
0.75	1.3717	1.0025	462.6	0.03517	0.600	0.0180
1.0	1.3744	0.85958	468.5	0.04279	0.600	0.0221

Table 2. Comparison of perturbation calculations of Wollkind and DiPrima (W & D) and present numerical calculations (F & D) for Type I disturbances.

Calculation	T	k_c	θ_c	R_c	c_{rc}
W & D	0.05	1.0207	-0.1022	5767	0.26403
F & D		1.021	-0.1026	5767	0.2635
W & D	0.1	1.0207	-0.2044	5750	0.26403
F & D		1.022	-0.2061	5750	0.2620
W & D	0.2	1.0207	-0.4088	5683	0.26403
F & D		1.027	-0.4234	5674	0.2553
W & D	0.3	1.0207	-0.6132	5571	0.26403
F & D		1.039	-0.6786	5519	0.2413
W & D	0.35	1.0207	-0.7154	5498	0.26403
F & D		1.052	-0.8577	5387	0.2263

Table 3. Energy integrals for critical values of the parameters for Type I and Type II disturbances with $E_v = 1$. The values of R_c, θ_c, k_c, and c_{rc} for a given value of T are given in Table 1. Numers in parentheses indicate the power of ten.

T	E_ϕ	D_ϕ	D_v	M	H	$D_\phi + D_v$	R(M+H)
			Type I Disturbance				
0.05	53.95	2312.	188.9	0.3990	0.03280	2501.	2490.
0.1	13.17	560.6	187.4	0.09701	0.03266	748.1	745.6
0.2	2.899	119.8	181.6	0.02096	0.03206	301.3	300.8
0.3	0.9513	36.98	169.8	0.006628	0.03082	206.8	206.7
			Type II Disturbance				
0.04	0.3475(-4)	0.3577(-3)	11.87	0.6334(-8)	0.1331(-2)	11.87	11.87
0.05	0.5482(-4)	0.5625(-3)	11.88	0.1265(-7)	0.1665(-2)	11.88	11.88
0.1	0.2177(-3)	0.2239(-2)	11.87	0.9926(-7)	0.3320(-2)	11.87	11.87
0.2	0.8664(-3)	0.8926(-2)	11.86	0.7778(-7)	0.6573(-2)	11.87	11.87
0.3	0.1956(-2)	0.2012(-1)	11.87	0.2607(-5)	0.9708(-2)	11.89	11.89
0.5	0.5420(-2)	0.5579(-1)	11.87	0.1148(-4)	0.1542(-1)	11.93	11.93
0.75	0.1213(-1)	0.1250	11.88	0.3546(-4)	0.2130(-1)	12.00	12.00
1.0	0.2139(-1)	0.2208	11.89	0.7567(-4)	0.2577(-1)	12.11	12.11

Legends for the Figures

1. Variation of R on the neutral surface with θ for $k = 1.02$ and for $T = 0$, 0.05, and 0.2.

2. Variation of R on the neutral surface with θ for $k = 1.02$ and for $T = 0.3$ and 0.5.

3. Variation of c_r on the neutral surface with θ for $k = 1.02$ and for $T = 0$, 0.05, and 0.2.

4. Variation of R_c with T.

5. Variation of c_{rc} with T.

6. Variation of θ_c with T.

7. Variation of k_c with T.

8. Type I eigenfunction for $T = 0.05$, $\theta = -0.1026$, $k = 1.021$, $R = 5767$. (a) ϕ_1 , (b) v_1.

9. Type II eigenfunction for $T = 0.05$, $\theta = 1.529$, $k = 1.370$, $R = 7132$. (a) ϕ_1 , (b) v_1.

10. Type II eigenfunction for $T = 1.0$, $\theta = 0.85958$, $k = 1.374$, $R = 468.45$. (a) ϕ_1, (b) v_1.

11. $U_1(z,\theta)$ for $T = 0.2$ and several values of θ.

12. Critical values of R as a function of θ for $T = 0.2$, $k = 1.02$:— — Orr-Sommerfeld equation; —— sixth order problem.

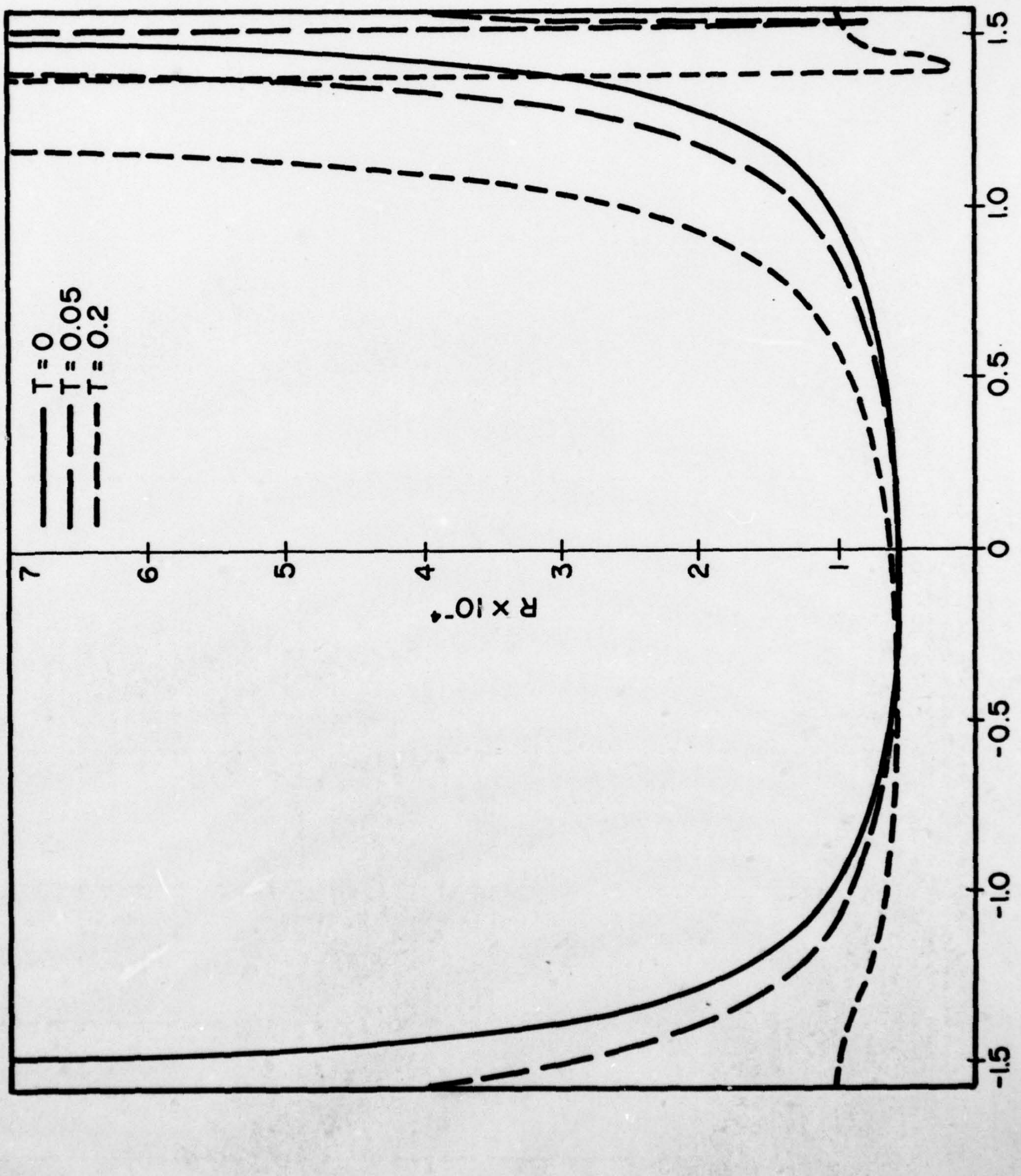


FIG. 1

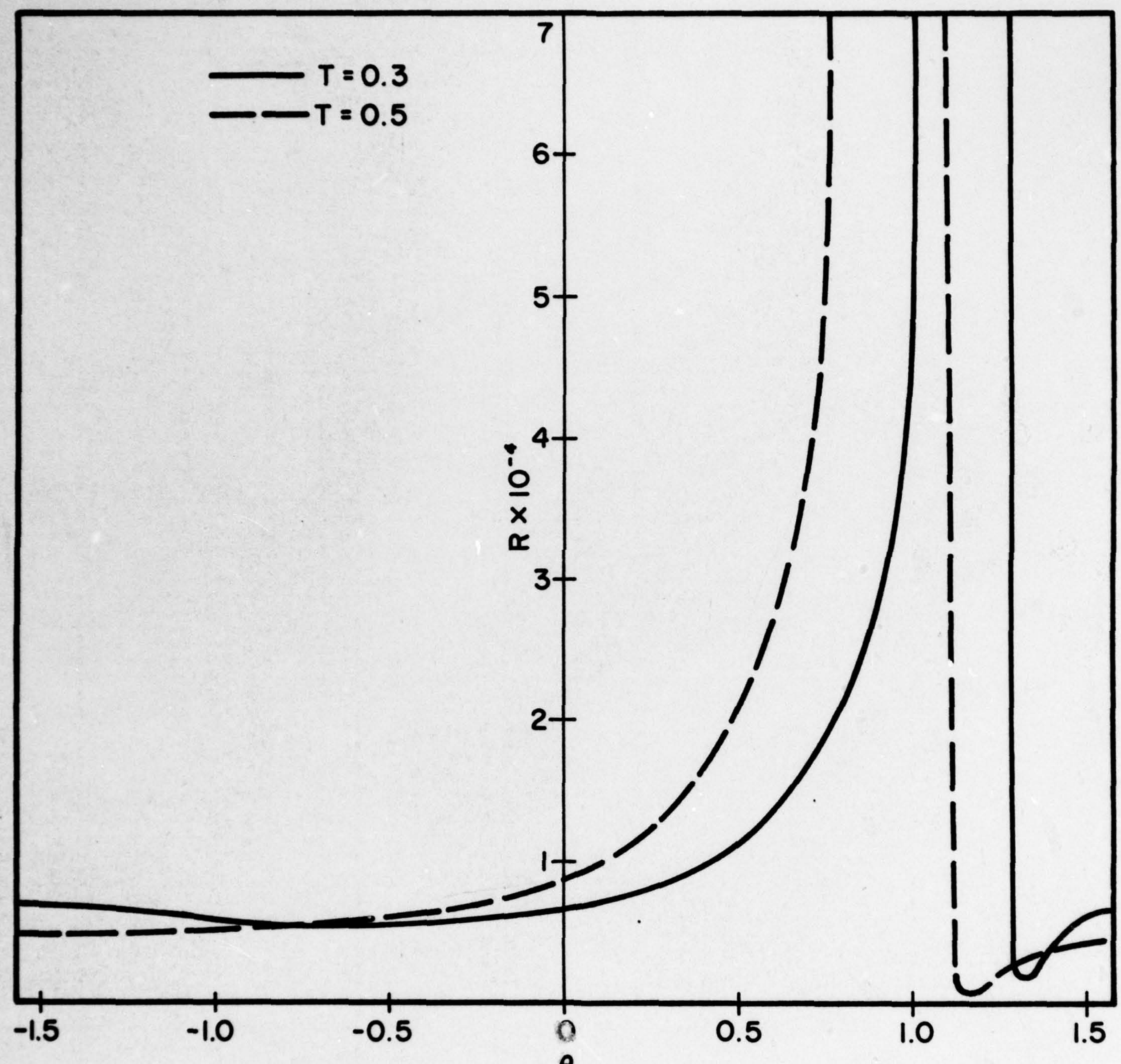


FIG. 2

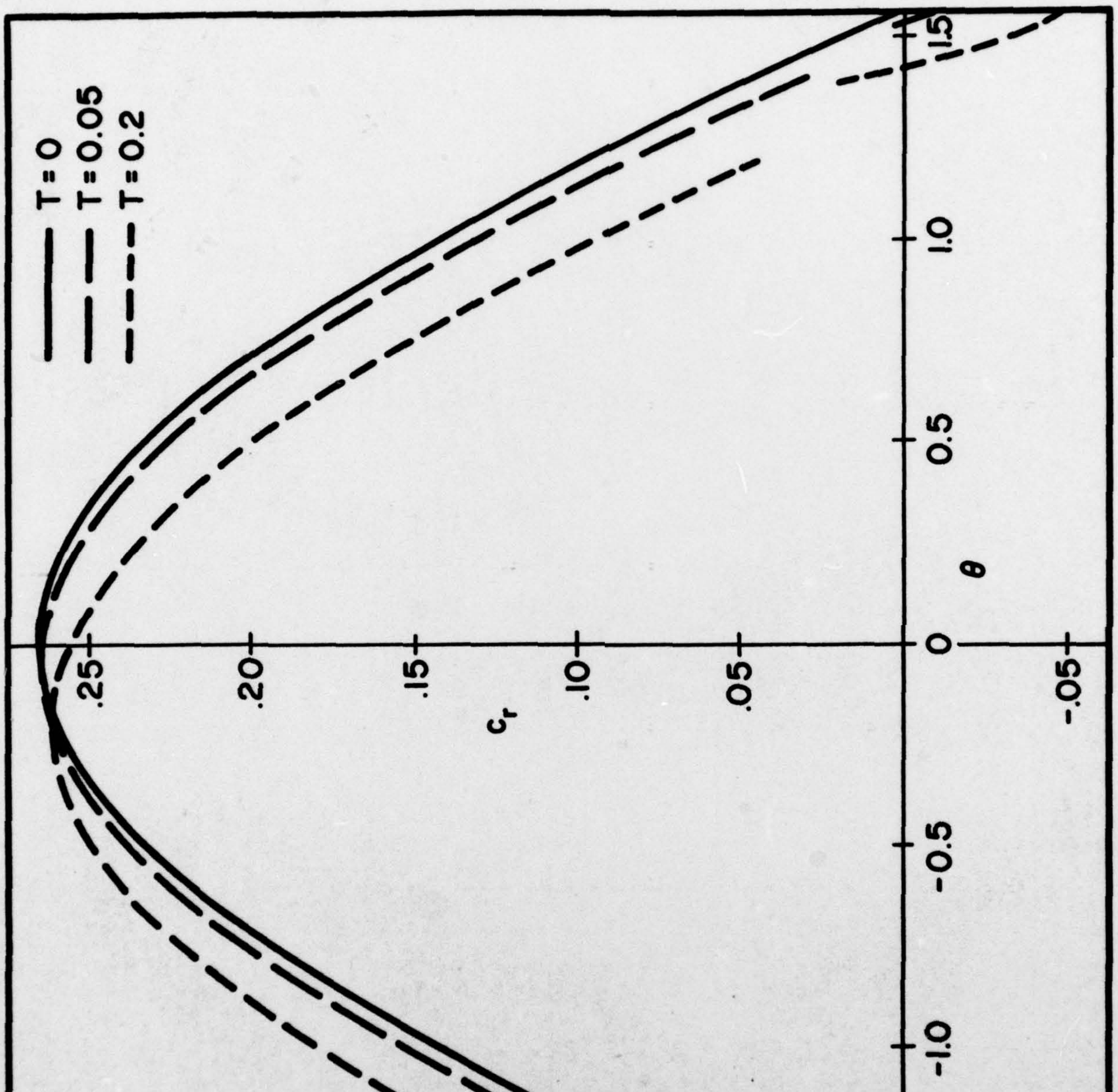
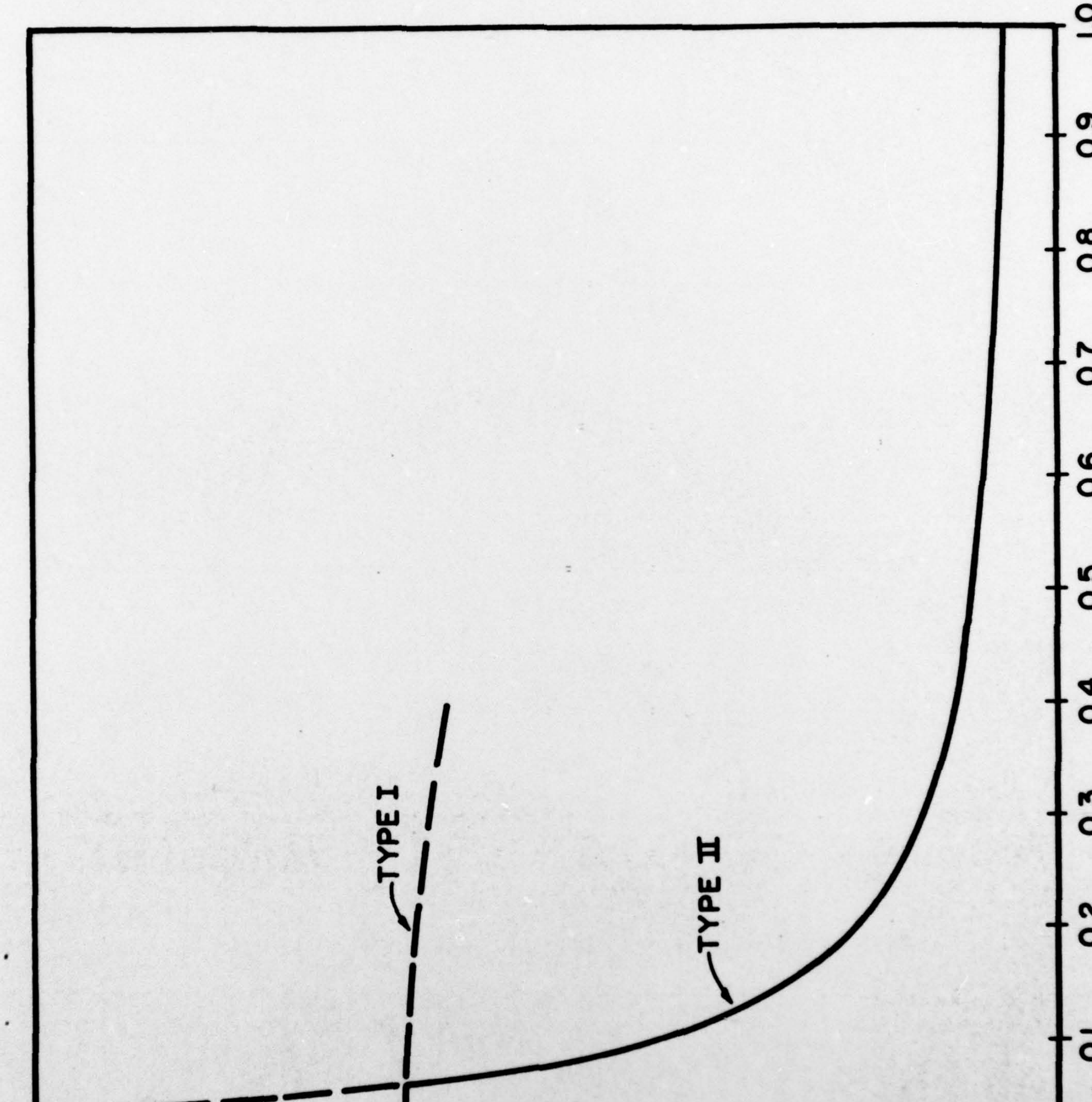
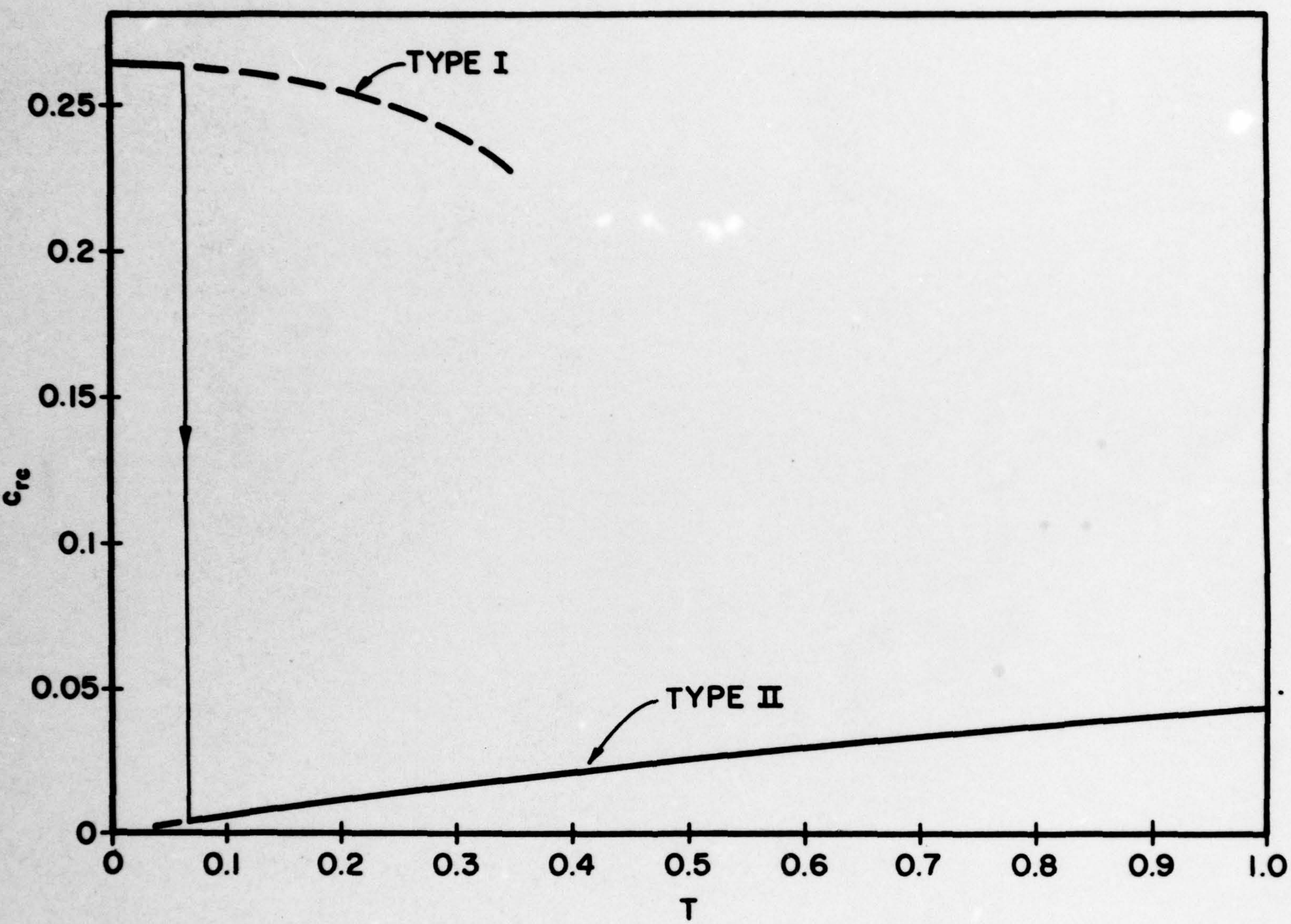


FIG. 3



TYPE I
TYPE II
T
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$R_c \times 10^{-4}$
0
1
2
3
4
5
6
7
8
9

FIG. 4



TYPE I
TYPE II
0.25
0.2
0.15
0.1
0.05
0
c_{rc}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
T

FIG. 5

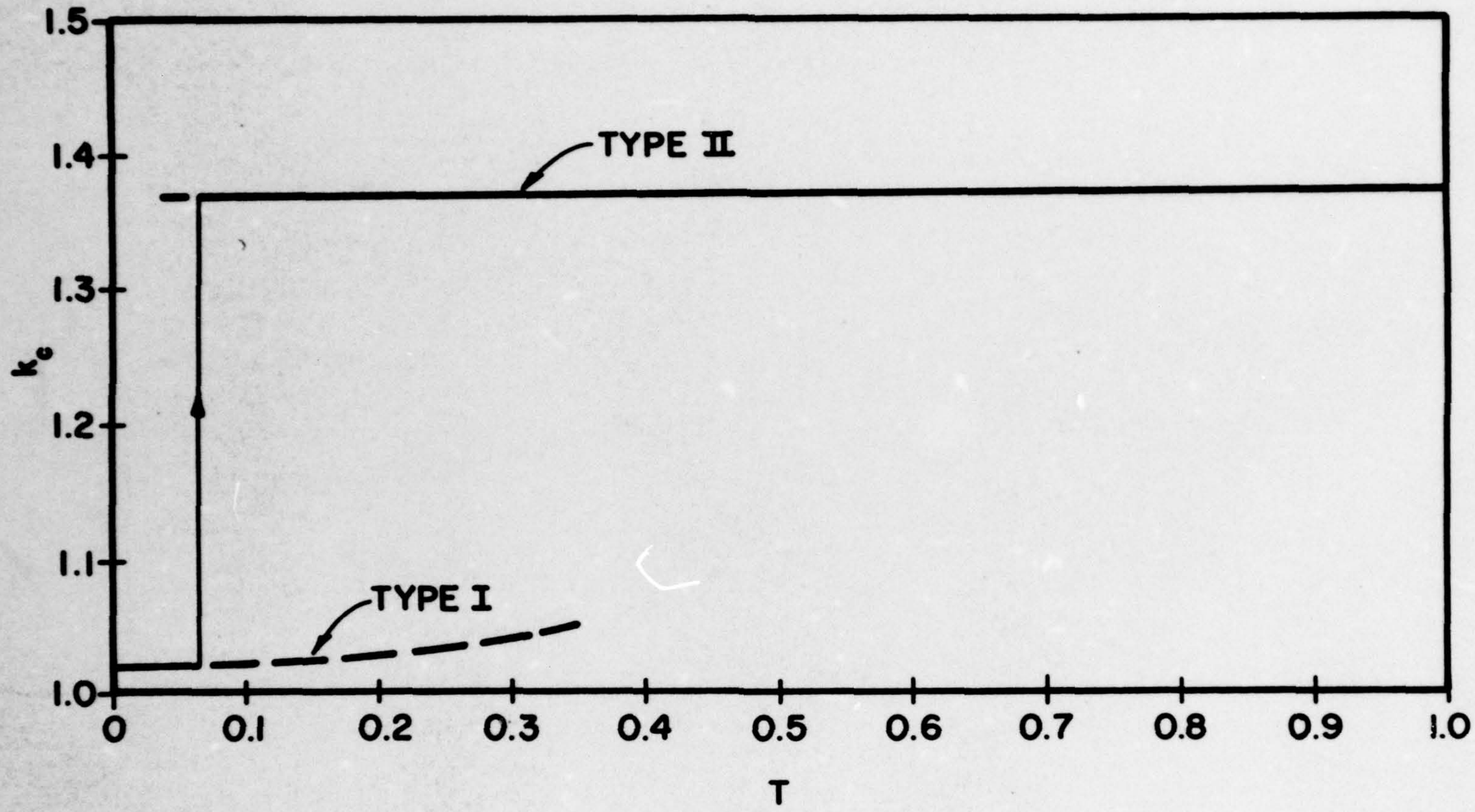


FIG. 6

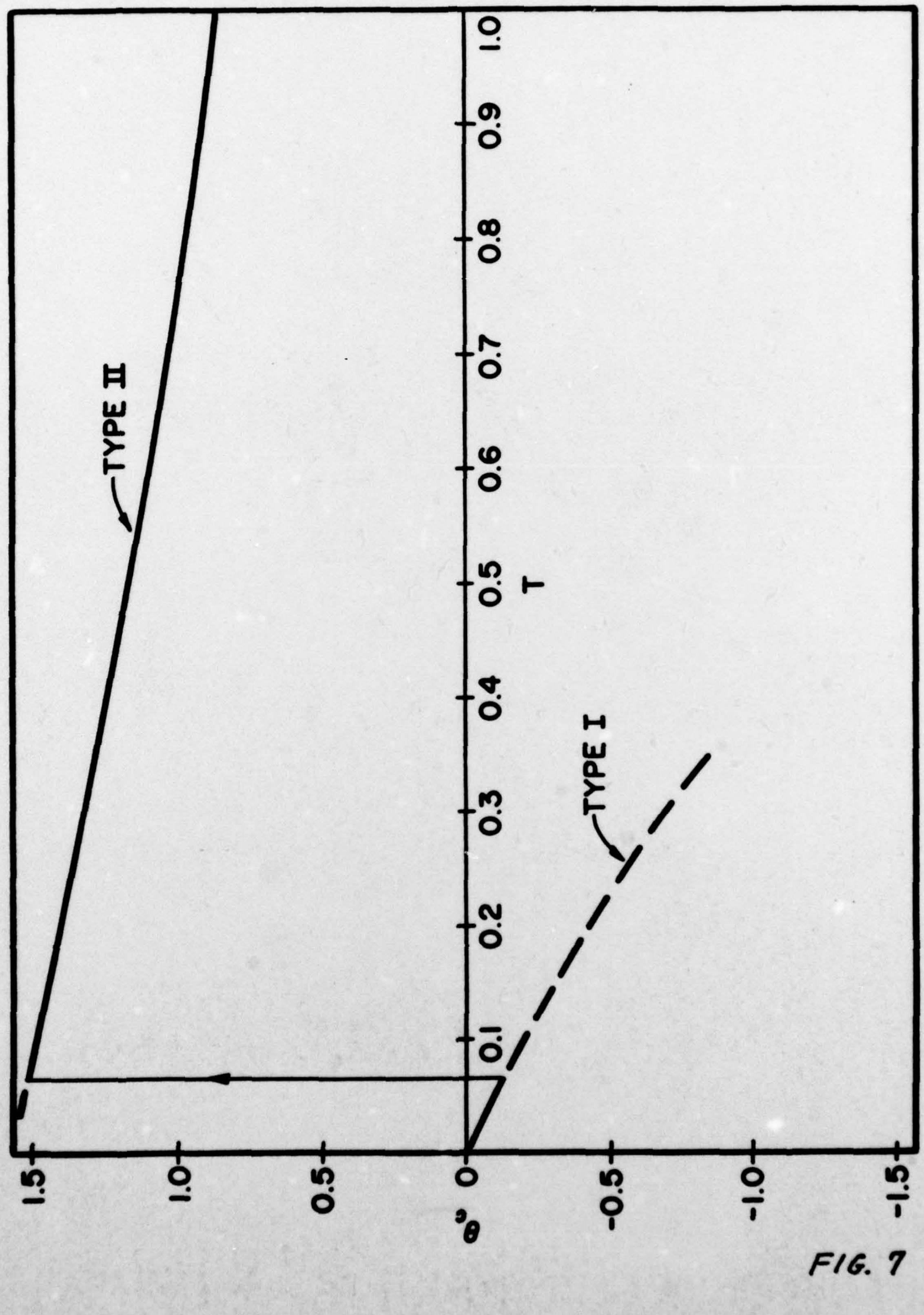


FIG. 7

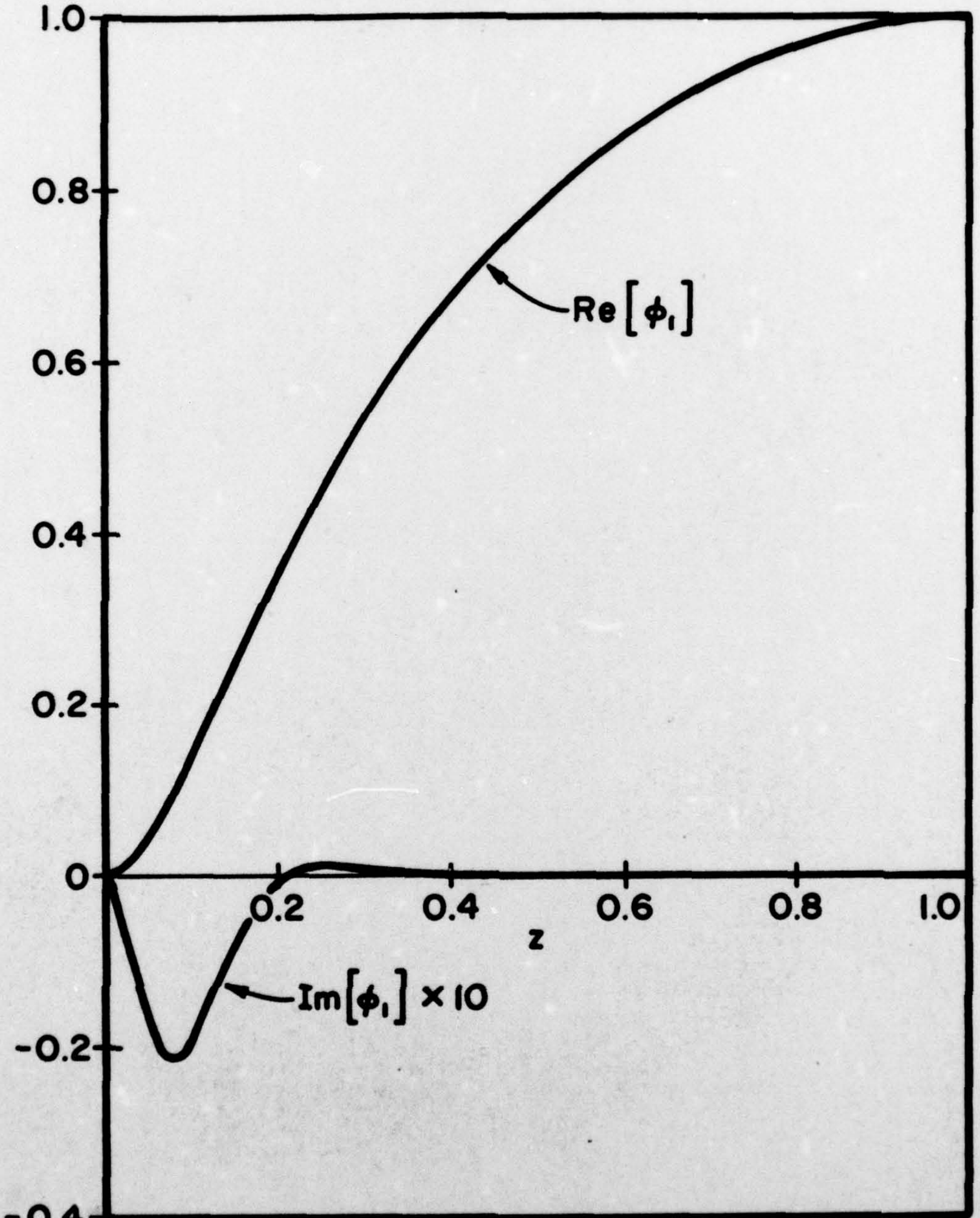


FIG. 8a

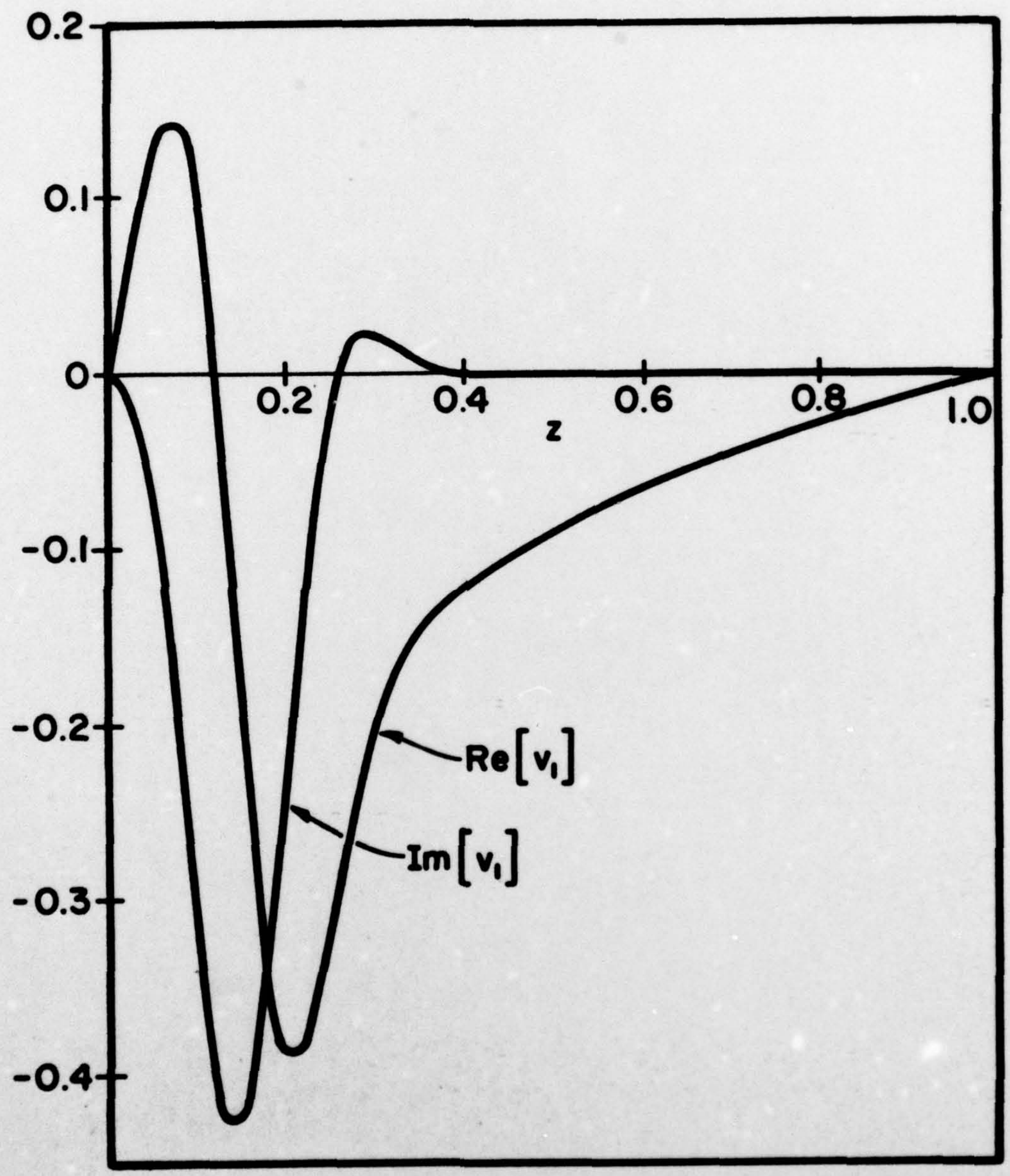


FIG. 8b

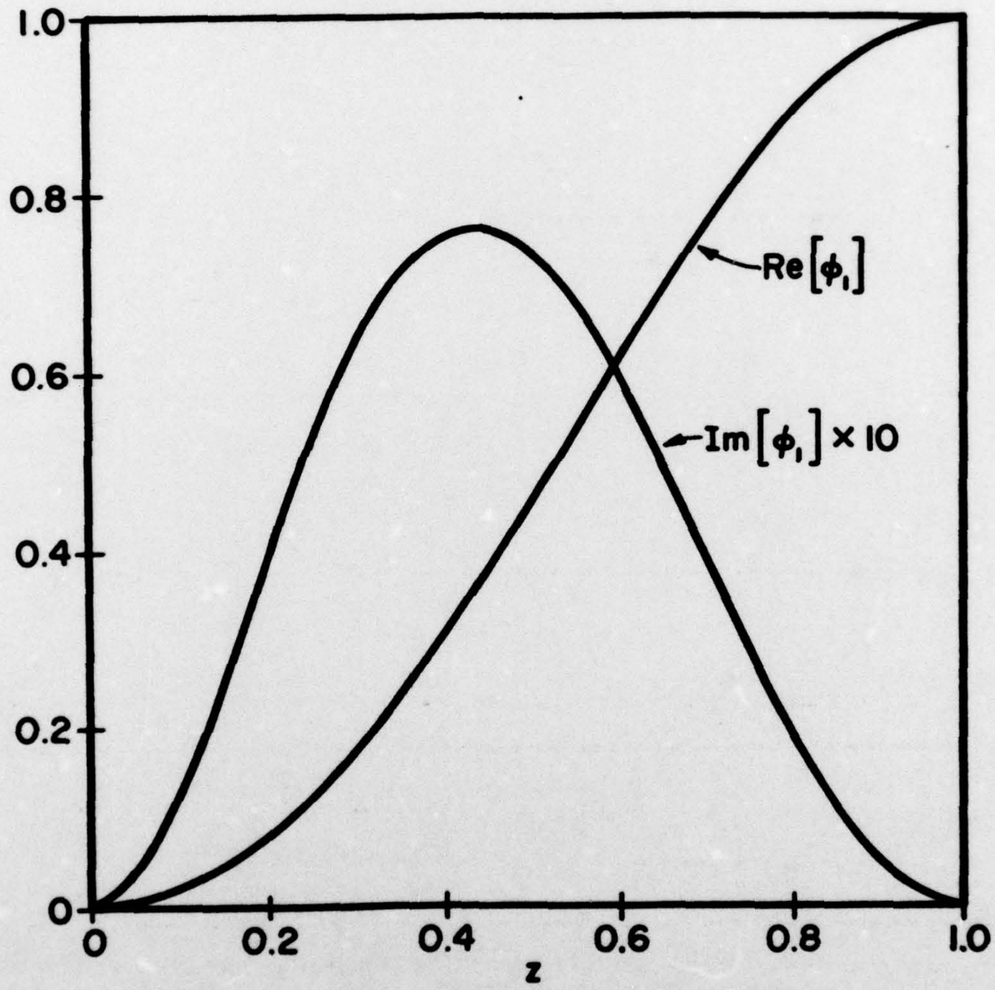


FIG. 9a

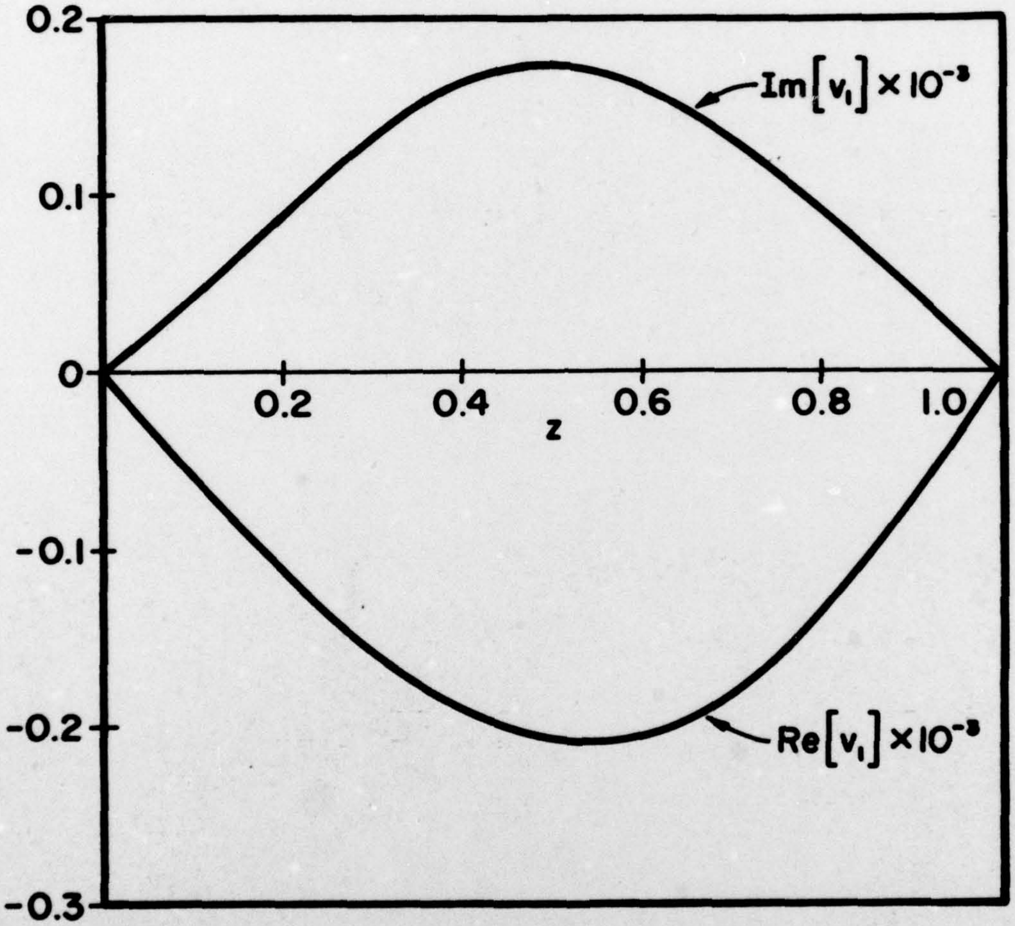


FIG. 9b

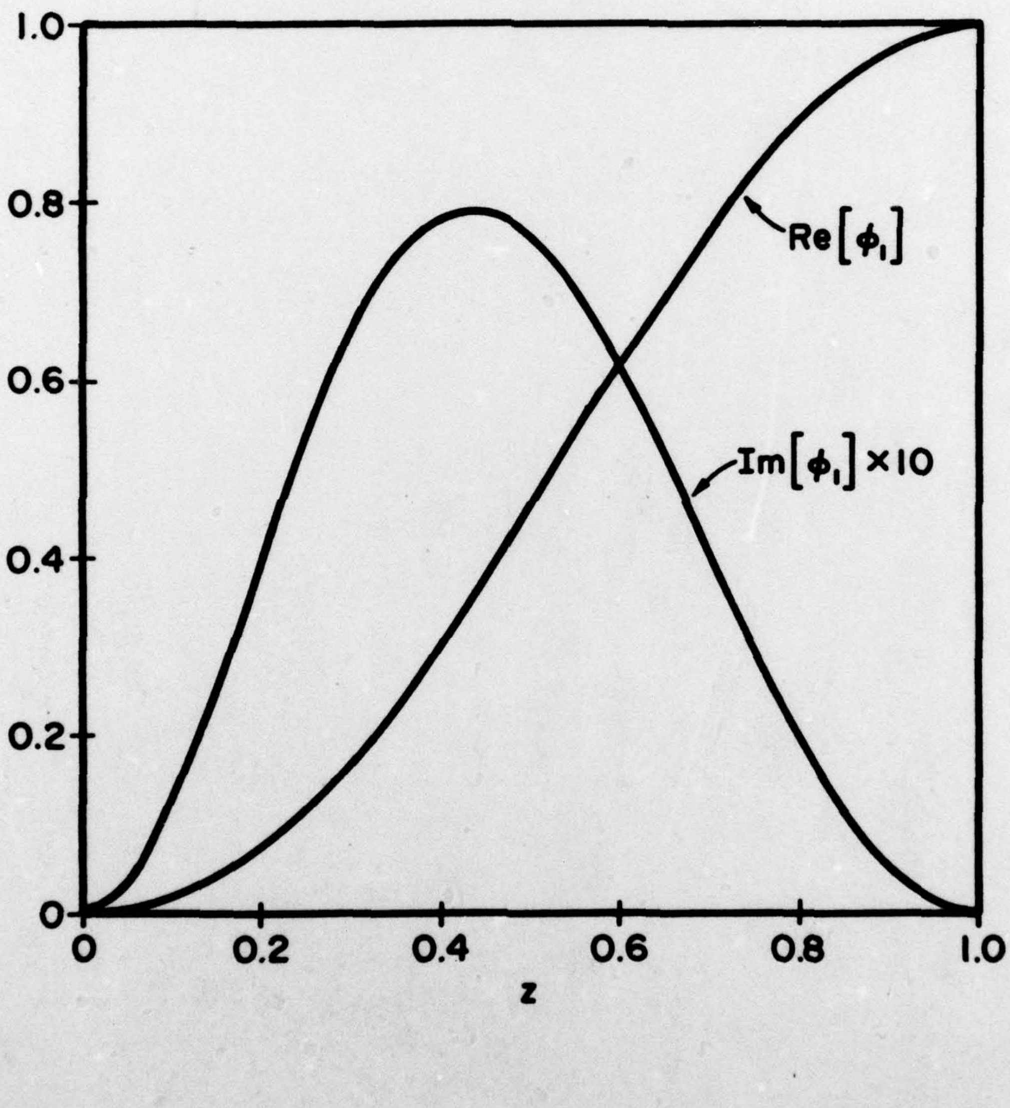


FIG. 10a

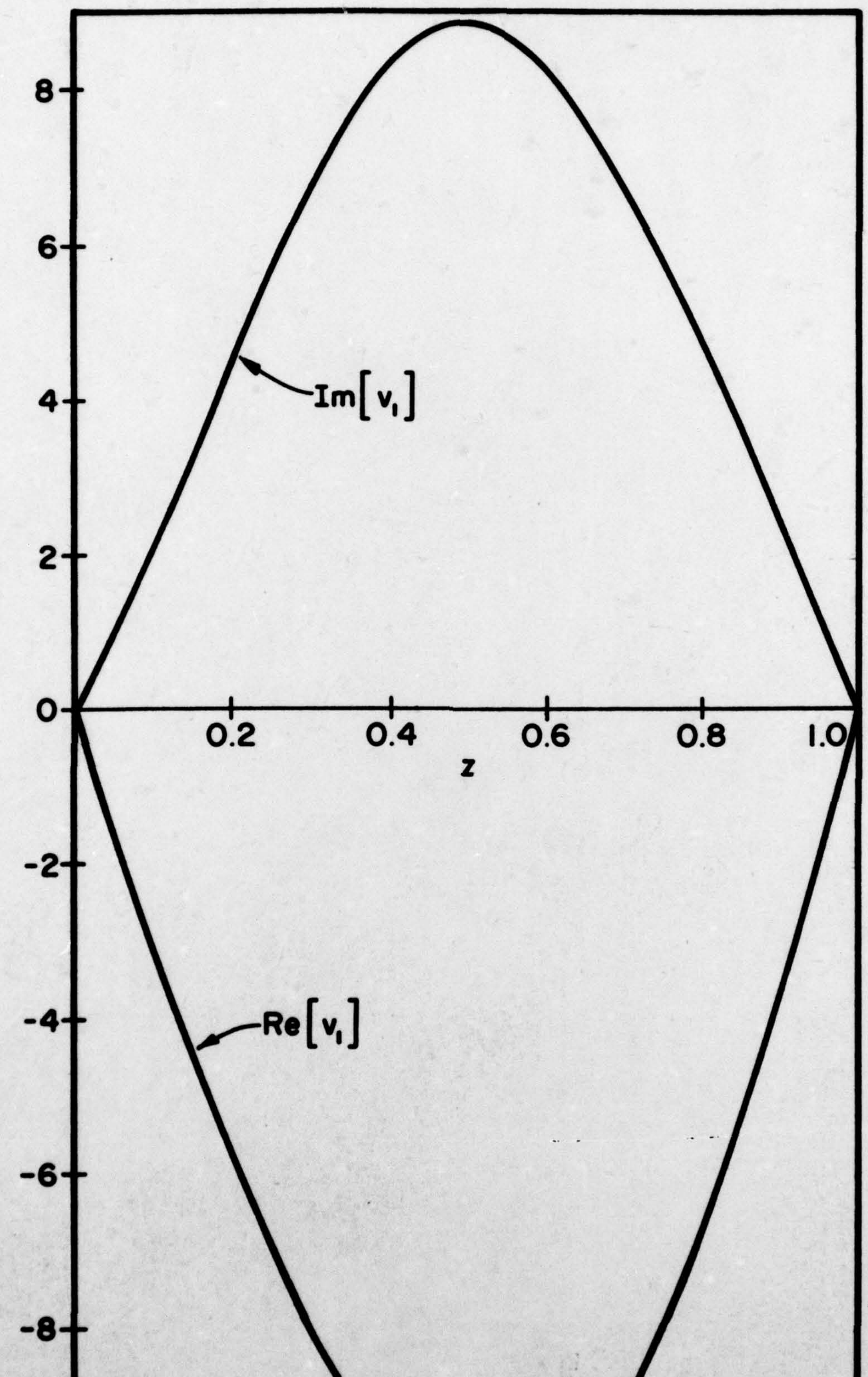
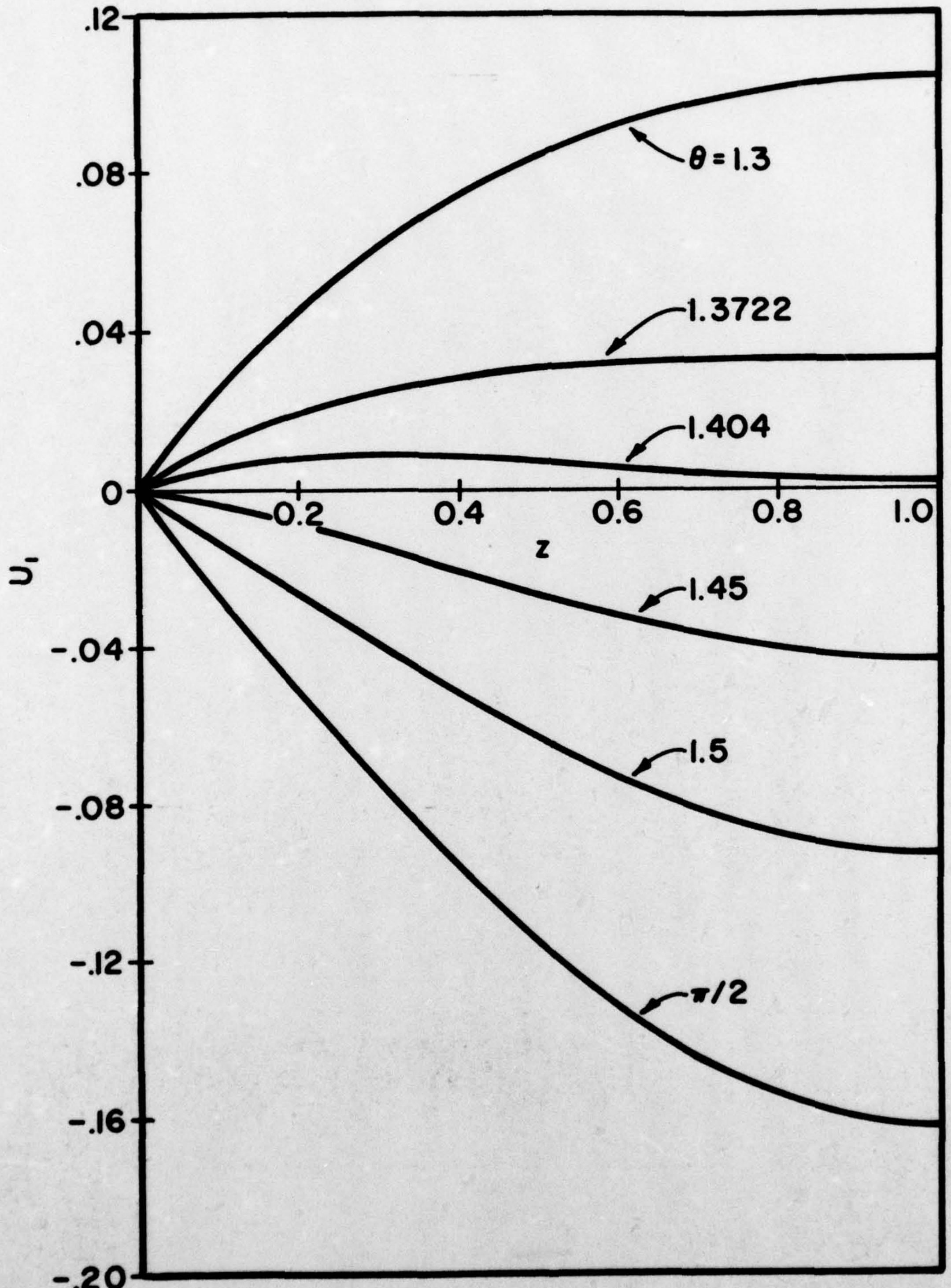


FIG. 10b



.12
.08
.04
0
-.04
-.08
-.12
-.16
-.20
0.2
0.4
0.6
0.8
1.0
z
U_1
$\theta = 1.3$
1.3722
1.404
1.45
1.5
$\pi/2$

FIG. 11

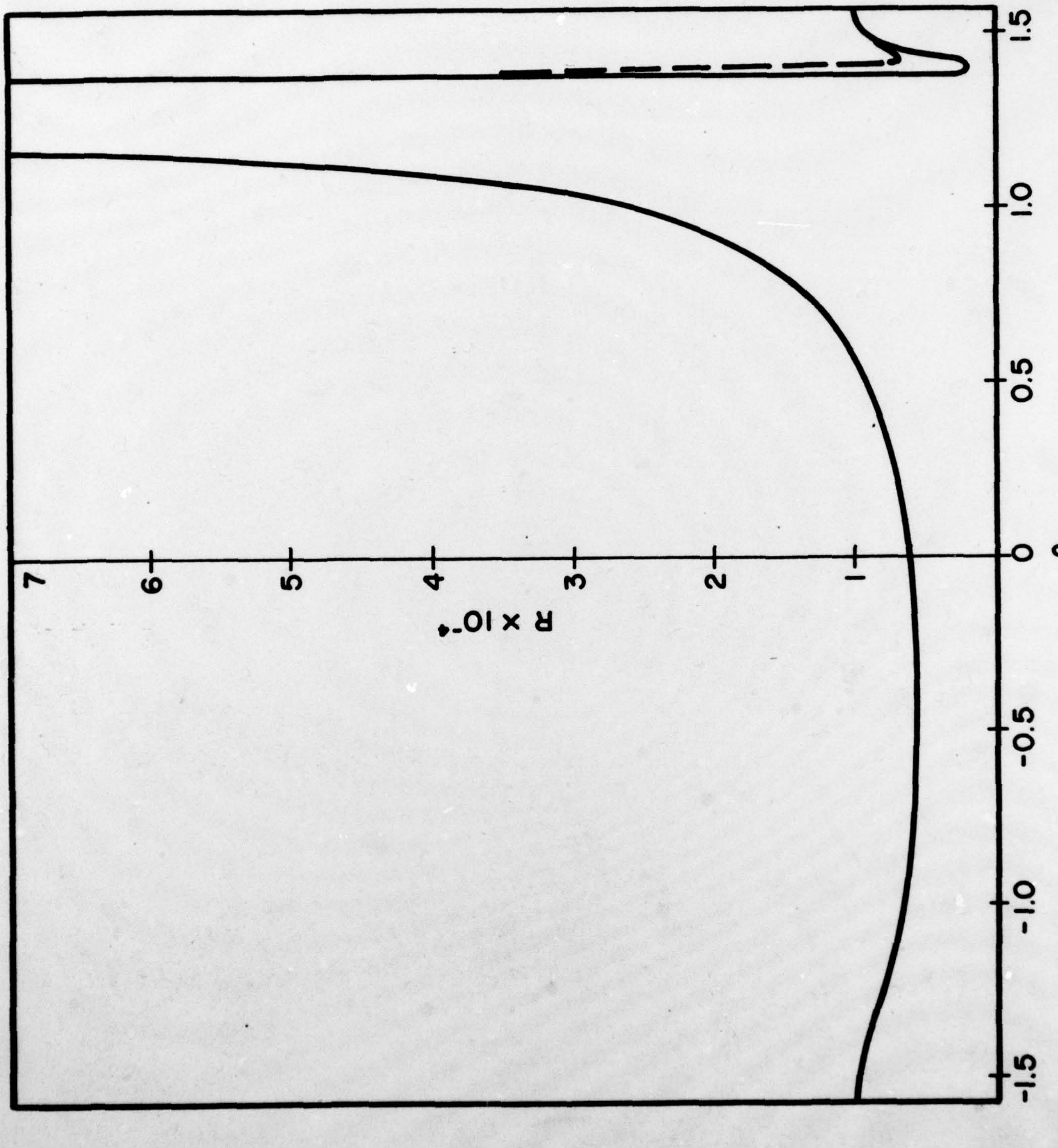


FIG. 12

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The stability of the viscous flow between two parallel horizontal plates due to a constant reduced pressure gradient in a system rotating about a vertical axis is studied. The critical value of the Reynolds number R, based on the reduced pressure gradient, is a function of a dimensionless rotation parameter T, the Taylor number. A numerical solution of the eigenvalue problem shows that (i) the viscous instability mode associated with plane

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Poiseuille flow at $T = 0$ disappears at a value of T of about $T \simeq 0.4$, and (ii) for $T \neq 0$ a new instability mode appears as a result of the Coriolis effect on the basic flow and in the perturbation equations. This new instability gives the critical value of R for values of T as small as 0.06.